Education Scotland
Foghlam Alba

Number and Number Processes

Professional Learning Resource

This resource is part of the suite of the Numeracy Professional Learning Resources

For Scotland's learners, with Scotland's educators
Do luchd-ionnsachaidh na h-Alba, le luchd-foghlaim Alba

Contents

Page

Introduction

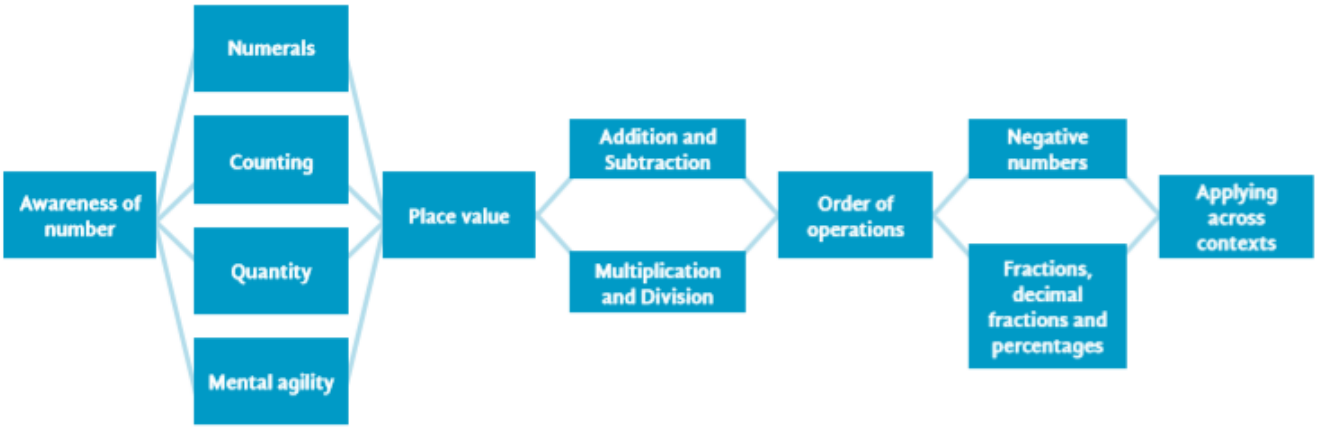
This professional learning resource has been created to enable practitioners to reflect on their own knowledge and understanding, highlight areas which children find challenging and outline effective approaches to support future learning and teaching in Number and Number Processes.

Numbers are all around us and they are used in many different ways. Developing an understanding of numbers and their role in the description of quantities is fundamental to forming the connections needed to describe a group of objects. To be confident and comfortable with numbers, it is necessary to understand how the number system works and how numbers relate to each other.

Developing an understanding that we have both words and symbols for all the numbers we use is important in developing an ability to count accurately and understand one to one correspondence. Learners need to visualise, hold and manipulate numbers as this helps to demonstrate an understanding of the number system and how it works.

Being able to apply numeracy skills across a variety of real life contexts leads to being numerate and being able to function responsibly in everyday life, contribute effectively to society and increase our opportunities within the world of work.

National Numeracy and Mathematics Progression Framework[1]
Number and Number Processes



[1] Further guidance on using the National Numeracy and Mathematics Progression Framework can be found by clicking here.

Early Level

The table below includes the experiences and outcomes related to 'Number and Number Processes' at early level. The experiences and outcomes should be used in the planning of learning, teaching and assessment. It is important to note that the benchmarks are designed to support teacher professional judgement in progress towards and achievement of a level. There are a range of different experiences that learners need to be exposed to before these can be achieved.

Experiences and Outcomes	Benchmarks
I have explored numbers, understanding that they represent quantities, and I can use them to count, create sequences and describe order. **MNU 0-02a** I use practical materials and can 'count on and back' to help me understand addition and subtraction, recording my ideas and solutions in different ways. **MNU 0-03a**	• Explains that zero means there is none of a particular quantity and is represented by the numeral 0. • Recalls the number sequence forwards within the range 0 - 30, from any given number. • Recalls the number sequence backwards from 20. • Identifies and recognises numbers from 0 to 20. • Orders all numbers forwards and backwards within the range 0 - 20. • Identifies the number before, the number after and missing numbers in a sequence within 20. • Uses one-to-one correspondence to count a given number of objects to 20. • Identifies 'how many?' in regular dot patterns, for example, arrays, five frames, ten frames, dice and irregular dot patterns, without having to count (subitising). • Groups items recognising that the appearance of the group has no effect on the overall total (conservation of number). • Uses ordinal numbers in real life contexts, for example, 'I am third in the line'. • Uses the language of before, after and in-between. • Counts on and back in ones to add and subtract. • Doubles numbers to a total of 10 mentally. • When counting objects, understands that the number name of the last object counted is the name given to the total number of objects in the group.

Establishing Strong Foundations for Learning

Careful consideration should be given to the **spaces**, **interactions** and **experiences** we provide, ensuring that opportunities for learners to develop the concept of number permeates across all.

Spaces

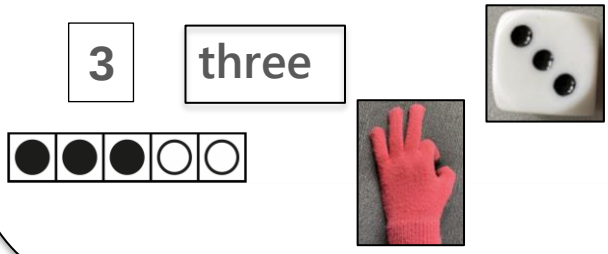
Rich, carefully considered learning spaces both outdoor and indoor can offer learners practical opportunities to develop the concept of number. The choice of experiences on offer should reflect an environment of open-ended possibilities in which children can feel intrinsically motivated to explore and investigate numbers through play. Selecting appropriate and engaging resources can enhance interactions; leading to creativity, curiosity and deeper learning.

Open-ended materials offer the potential for creative explorations through child-initiated and adult initiated learning experiences. Spaces should be planned to provide a balance of opportunities for learners to play, explore, investigate and question. Practitioners should ensure that planning for learning starts with the child and is carefully balanced to be both responsive and intentional in design. Opportunities should enable learners to make sense of numbers and number processes in the world around them, whilst also ensuring learners' needs are being met through their engagement with the experiences and outcomes presented within early level.

There are many different ways of permeating number across the learning spaces. Some examples of how to do this are provided below.

Typed and handwritten digits and numerals prominently displayed.

Signs which show numbers displayed in different ways.



Display numerals from the wider environment such as door signs, bus numbers and road signs.

Provide containers of different sizes and capacities.

Provide access to real life objects containing numbers. Items could include calendars, diaries, calculators, tape measures, price tags, recipe books, shopping lists, measuring scales, telephones, coins and many more.

Provide a variety of types of containers such as egg boxes, tins, baskets, cake trays, muffin tins.

Provide a variety of loose parts such as buttons, corks, pasta, shells, pegs and beads.

Provide sets of items such as cups, bowls and plates.

Provide collections of toy cars, people and animals.

Provide a variety of natural items such as leaves, stones, twigs, pine cones and conkers. These can also be collected by the learners themselves.

Provide a range of mark making materials such as chalk, brushes, markers, clip boards and paint.

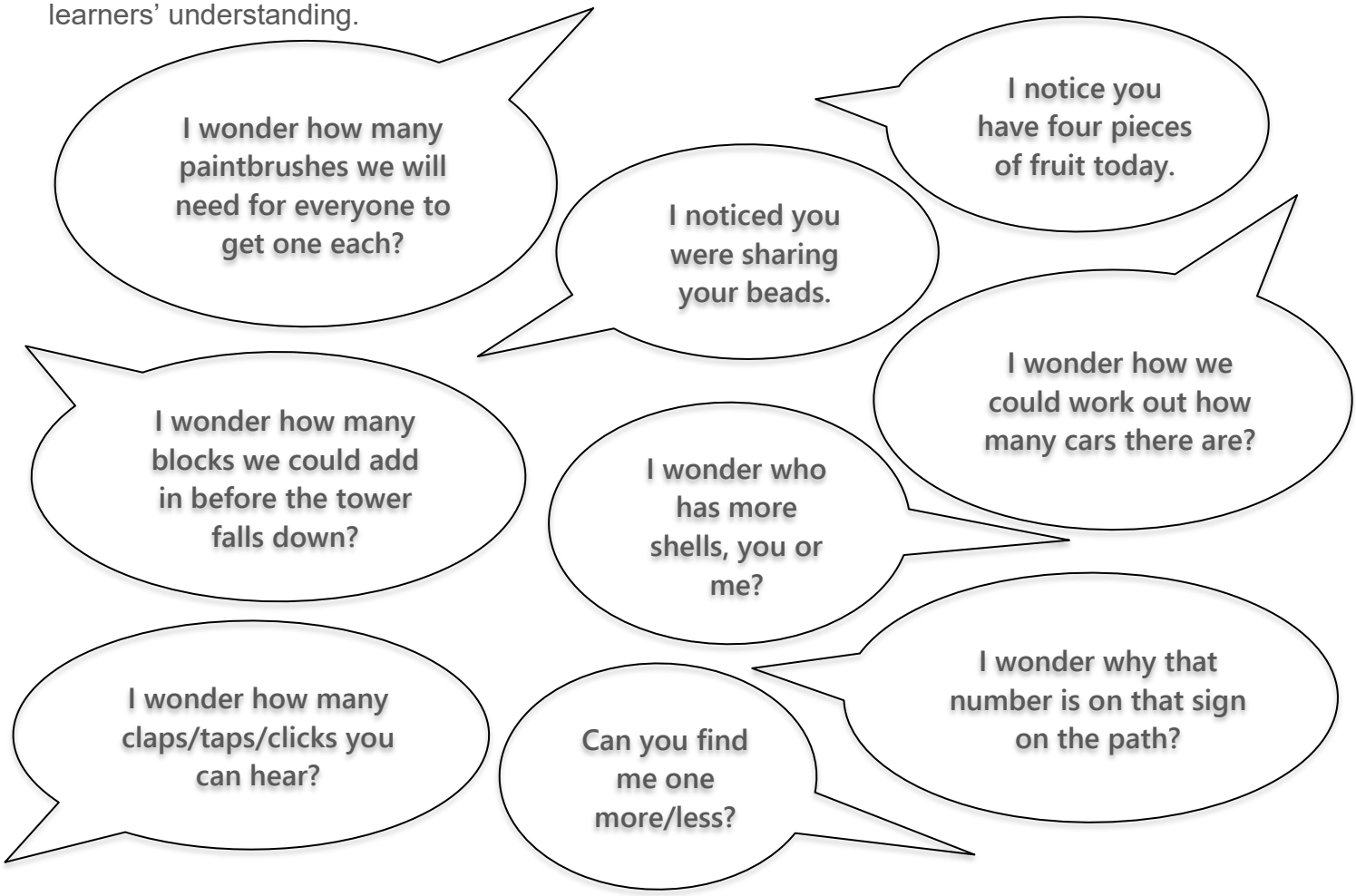
Provide richly illustrated story books with representations of number.

Provide malleable material such as dough, modelling clay and associated tools.

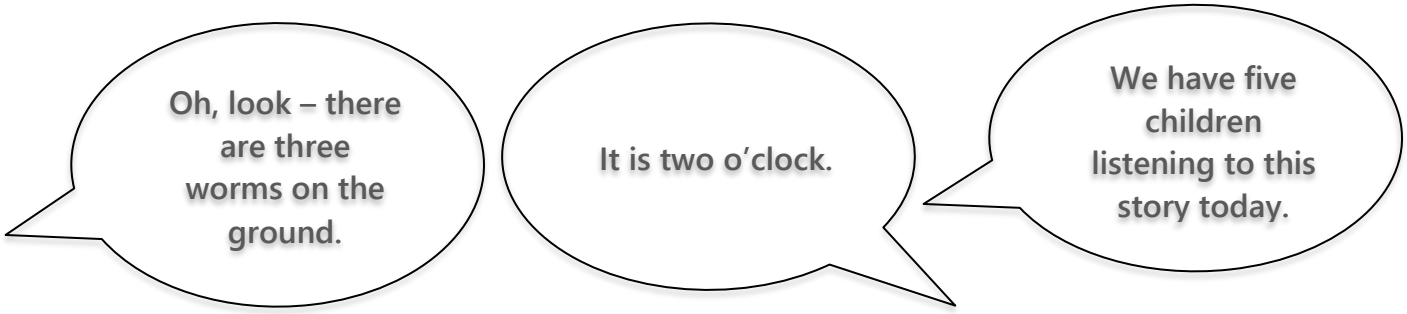
Interactions – One of the roles of the Practitioner

One of the roles of the practitioner is to determine what the child could learn through their own interests using high quality interactions. Practitioners should support learners to extend their learning of number and number processes through encouragement of creativity and curiosity. Careful observation is an important assessment tool, as is knowing when to stand back and give learners time to investigate by themselves. It is important to notice when learners are more receptive to further support from the practitioner and address any misconceptions that may cause challenges at later stages.

Language can be used as a powerful vehicle for the teaching and understanding of number and number processes. When interacting with learners, practitioners can model the use of relevant vocabulary in meaningful contexts. This allows learners to make links between the spoken word and the associated learning. Using language in the correct context regularly will support learners' understanding.



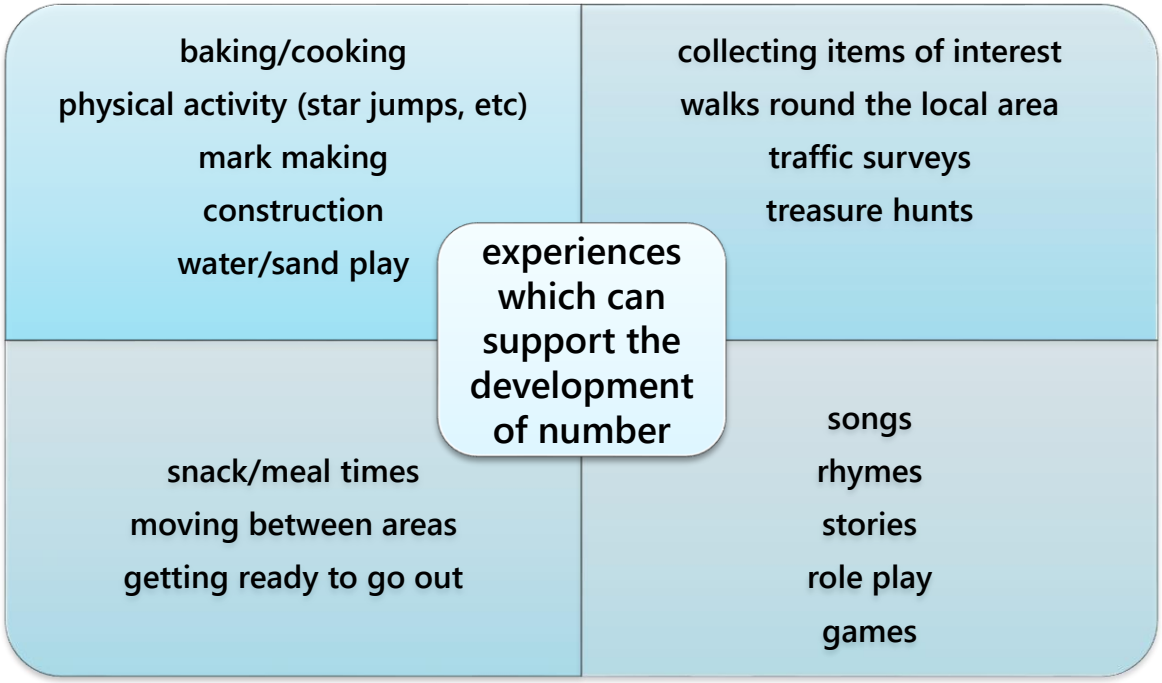
Practitioners should also embed numbers in everyday conversations.



Experiences and Routines

Learners need to be immersed in number. This includes numerals, associated number names and corresponding quantities for each number. Providing strong foundations and developing number sense at the early stages of learning are important as these skills help support young children to positively engage in and investigate a range of basic mathematical concepts.

Experiences of everyday activities are important, both indoor and outdoor. Some examples of these are noted below, there are many more.



Within these experiences the following number skills can be developed.

- subitising
- identifying the number before and after
- understanding concept of zero
- using ordinal numbers
- identifying the missing numbers in a range of sequences
- numeral recognition/identification
- recognising different ways a number can be represented (see page 5)
- counting: 1:1 correspondence, stable order, cardinality, abstraction, conservation of number, order of relevance[2]
- forward and backward number word sequences

[2] Counting Principals: Gelman and Gallistel (1978): Gelman, R. & Gallistel, C. (1978) The Child's Understanding of Number. Cambridge, MA. Harvard University Press.

Stories, Rhymes and Songs

In ELC and early primary we know that stories, rhymes and songs connect young children to many aspects of learning and development through diverse, meaningful and rich contexts. Their words and rhythms provide a fun and creative way of exploring numbers and number processes alongside developing associated vocabulary. Incorporating these into practice also enhances movement, coordination, positional language, as well as supporting early number sense. Our brains have become hardwired to respond to stories and learners can use stories, songs and rhymes to make sense of the world. Their repetition reinforces learning through encouraging a young child's natural schematic behaviour.

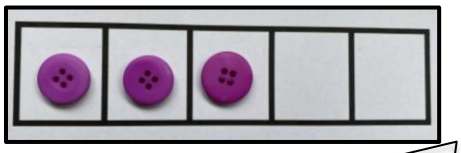
Through a range of traditional and contemporary songs, stories and rhymes we can engage learners' interests and curiosity and help them understand abstract concepts. There are many stories, rhymes and songs that address mathematical concepts and they can be adapted or used in provocations to illustrate a concept or pose an investigation for learners to problem solve.

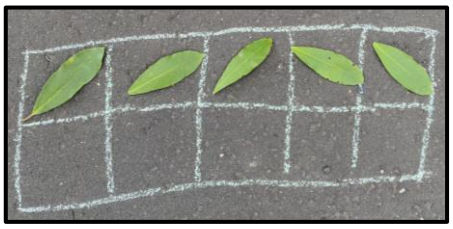
ELC and early primary settings already provide a rich array of songs, stories and rhymes for young children. Further inspiration, if needed, can be found on the Scottish Book Trust's Bookbug pages.

We already know that actions, puppets and objects are effective in acting out what is happening in the stories, rhymes and songs – but consider how these also provide young children with visual and physical reinforcement of concepts. Encouraging young children to be creative with stories, songs and rhymes can provide a powerful observational tool in assessing their understanding of concepts.

Number Word Sequences and Numerals

Five and ten frames are excellent resources for developing number sense within the context of ten. They can be created using physical objects and materials that the learners are interested in and familiar with.





How many buttons have you collected?

What do you notice?

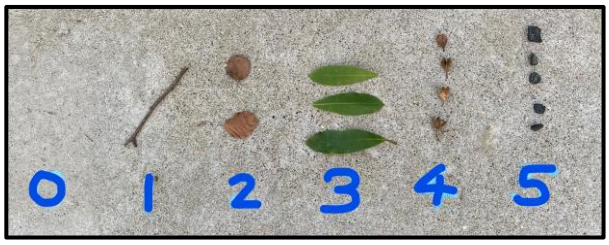
I wonder if there is a different way we could show this number?

I think that there are five leaves. How can we check?

I wonder how many stones there are?

Learners could be asked to find numeral cards that match the numbers.

It is important to develop numeral recognition. Adults should model numeral formation daily and display a range of written and typed numerals. Number lines can provide a visual support in developing an understanding of number.



Can you find the number five?

Learners could create giant number lines that they can stand on.

Cover a number on the number line with a beanbag. 'I wonder which number has been covered? '

I am standing on the number four and you are standing on the number six. Which number is in between us?

Counting Collections

The important skills and understanding that make up counting can be developed through many opportunities to count; this could be developed through using counting collections. It is always important to start from where the child is at in their learning and development. This is determined through close observation.

Learners should be provided with a variety of interesting materials to count; the collection size should vary according to learner need and/or specific learning intention. The collection could involve counting counters, straws, stones, toy cars, keys; essentially any objects available or of interest to the learners.

Leaners could either work alone or together to determine the total; negotiating the way they will count their collection and then recording how they counted. Learners can be encouraged to count and represent in ways that make sense to them. As learners count, they might move objects in and out of containers, set them out in a line, sort into groups by colour or size, etc. Through practise, modelling and scaffolding, learners develop more efficient and accurate methods. Constant observation of how the child interacts with the materials is crucial to determine how their thinking is developing and what can be provided to reinforce or further develop learning.

Learners are working to coordinate three aspects of number during a counting collection activity. For example, to really understand what 6 means, it is helpful if learners are provided with opportunities to connect the name (“six”), the quantity (6 items), and the written numeral (6).

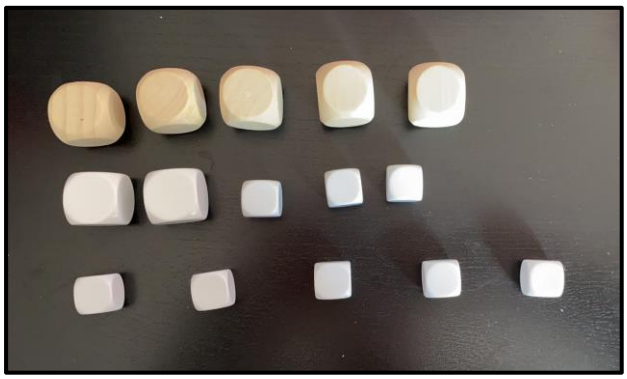
It is important to develop understanding of the fact that anything can be counted. Opportunities to count non-physical objects such as claps, beats of a drum, environmental sounds or words in a sentence are a good way to explore this concept. Experimenting with sound patterns at different speeds can develop this understanding further, for example, 5 slow claps compared with 2 quick claps and 3 slow claps.

Lots of exposure to hands on activities using numerals will help consolidate learning. Some examples of these are listed below.

- Using sensory art to create a number using sand, paint, foam, etc.
- Going on a ’number hunt’ matching number names and objects of the same value.
- Creating a numbered car park with numbered cars, and suggesting children to place, for example, car 8 in parking space number eight at tidy time.
- Having numbers and number names on containers. Children select a container and put objects of their choice in the container to match the number.
- Using numerals to label spaces where resources are kept.

Points to consider:

- Learners need plentiful opportunities to develop the skill of cardinality; being able to recognise the final number of count as the cardinal value.
- Some learners may assume that when objects are spread out that there are more of them (conservation of number). In the picture below, the blocks are spread out differently but there are still five in each row.



- When counting, learners may use number names in an incorrect order, or may count objects more than once.
- Some learners may appear to be counting confidently but that may be because they are just be mimicking the rhythm of the counting pattern.
- Some learners may not understand the relationship between numbers and their associated quantities.
- Learners may mix up certain numerals, for example, 6 and 9 or 12 and 20.
- The concept of zero can be challenging for some learners. Songs, games and activities can be used to reinforce that zero means ‘nothing’.
- Children will need lots of opportunity to count combinations of objects and to recognise totals without counting (subitise). Using familiar dot patterns such as dominos and dice helps children to visualise what a number looks like without having to count from one.

Addition and Subtraction - Effective learning and teaching approaches

When introducing addition and subtraction it is important that the experiences provided are relatable to the lives of learners. When children are confident counting a group of items and have a sound understanding that the last number they count is the total they should then be given lots of opportunity to count sets of objects together. Lots of practical investigations using concrete material are essential before formal calculations are introduced.

It is important that addition and subtraction are taught together as this will support a deeper understanding of the relationship between them.

Toys, food and other everyday objects can be used to develop understanding of addition and subtraction.

I wonder what would happen if I added two more cars?

I wonder what would happen if I took away two cars?

You have three cars and I have two cars, how can we find out how many we have altogether?

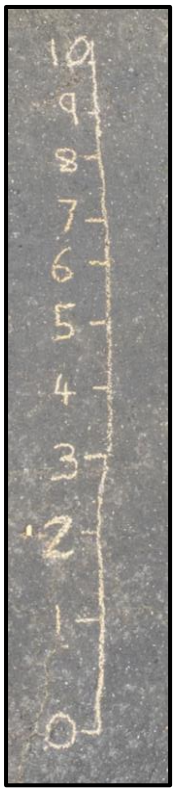
This could then progress to screening one of the sets to develop the learners' skills in counting on from different numbers. In the example below the first set of cars have been covered with a sheet of paper.

There are three cars under the paper. I've added two. How many do we have now?

I had some cars. I've taken three away, now there are two. I wonder how many cars I started with?

Backwards counting is a fundamental building block for learners enabling them to build skills in counting back and subtracting accurately.

Previous suggested activities in relation to number lines can be extended to develop the concept of addition and subtraction.



I am on the number five and you are on the number two. How many jumps will it take you to get to me?

I wonder what number you would be on if you jumped backwards one space, two spaces, etc.?

I wonder what number you would be on if you jumped forward one space, two spaces, etc.?

Learners can use dice and dominoes to develop their understanding of addition and subtraction.

I see a four and a one

I see 3 + 2

I see two plus two plus one

Let's roll another die. I wonder how many dots do we have now.

What do you see? How do you see it?
I wonder if you can show me the total number of dots in a different way?

Finger Patterns

Learners learn to make sense of the numbers around them in lots of different ways. This can often include using their fingers to help them count. This is a very important stage in learners' number development and should be actively encouraged and supported.

Children will initially count their fingers in sequential order 1, 2, 3 before being able to recognise the different finger patterns. It can take some learners time to realise that they have 5 fingers on both hands making 10 altogether. It is important to spend time teaching finger patterns to help learners develop this understanding.

Watch me as I count my fingers. Can you do this too?

I have made four, can you do this too?

I am going to use my fingers to make doubles. 4 + 4 = 8

I wonder what happens if I add one more finger?

I have five fingers on this hand and two more on this hand. I wonder how many altogether?

When asked to show four fingers, learners generally start with counting one at a time; one, two, three, four before proceeding to show four. With practice, this can be progressed further as learners show four fingers automatically, without the need to count individually. To provide further challenge, learners can be asked, "I wonder if you can show me four using two hands?"

When children can confidently make finger patterns without counting they can be asked to hide their fingers either raising them above their eye line or behind their back to make tasks more challenging.

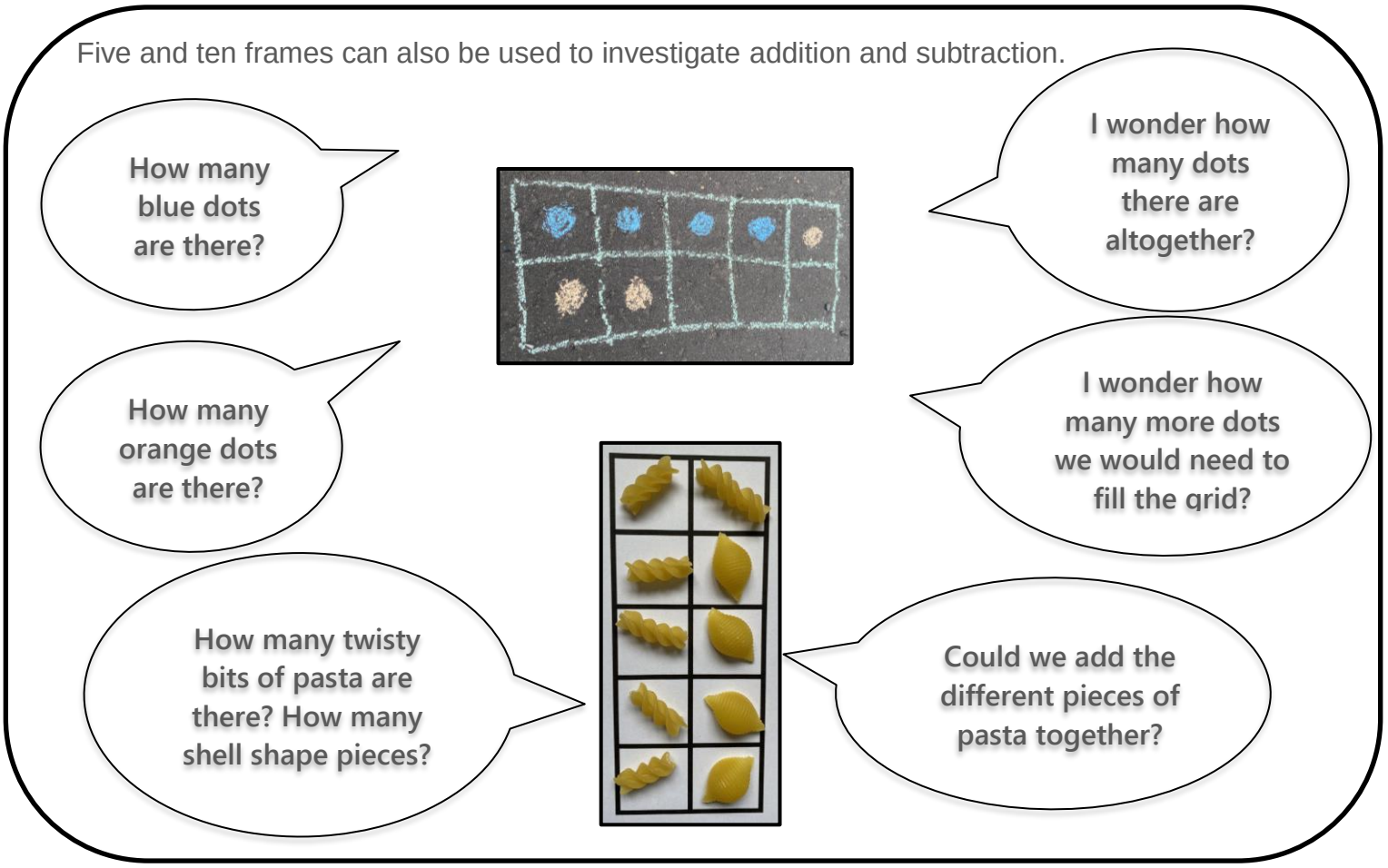
Put your hands behind your back. Can you put up six fingers? How many fingers are down?

Without looking at your fingers can you show me seven?

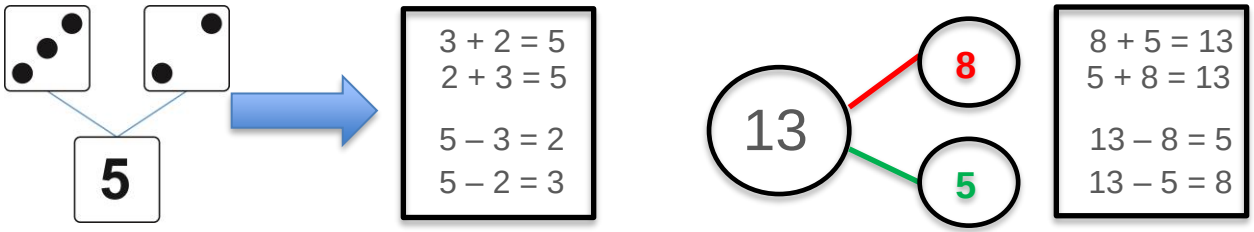
Put both your hands above your head. Can you show me eight fingers? Can you show me this in a different way?

Some learners may need additional support when using finger patterns due to their motor skills and finger dexterity. It is important to consider how we will adapt activities to ensure all learners can take part. It may be that the learner uses the adult's fingers or they could use electronic hands on a smart board/tablet. Another way to adapt the activity is to use waterproof gloves filled with sand.

Five and ten frames can also be used to investigate addition and subtraction.

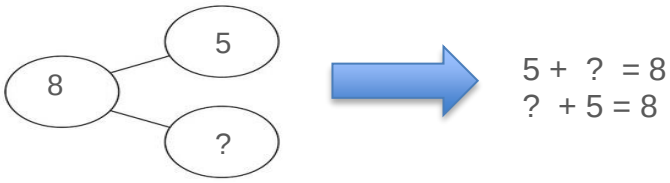


The use of concrete materials can then be extended to develop written calculations to support understanding of the part-part-whole approach; allowing learners to see the relationship between a number and its component parts. For example, ask learners to roll two dice and record the numbers they see. “Can you make a number sentence using the two numbers you have rolled”? “I wonder how many dots we have altogether?”

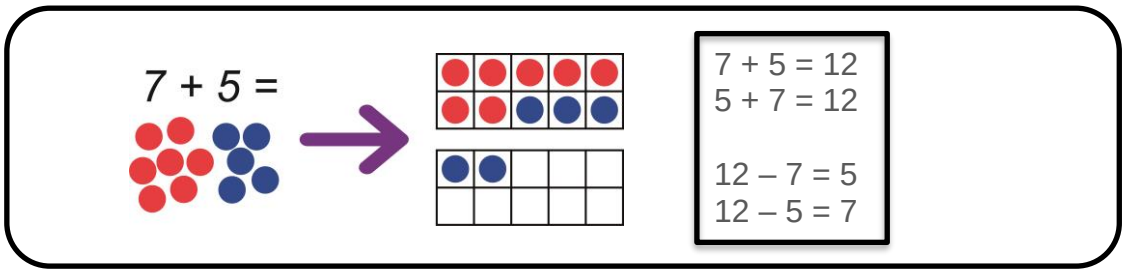


This can lead to learners being able to develop an understanding of the relationship between the whole number and the component parts and can support the understanding of the connection between addition and subtraction.

What is the missing number? How do you know? Can you prove it?



Using ten frames is another useful concrete material which can help facilitate this thinking. These can be physical ten frames that learners can place items on or into and can progress to pictorial representations.

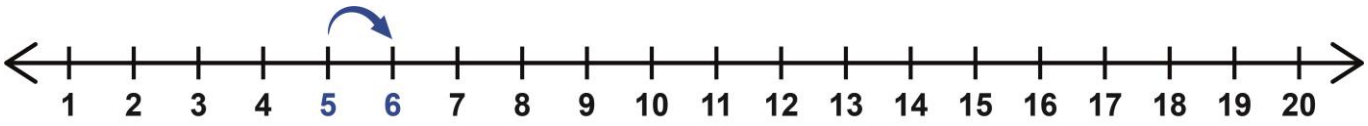


Learners are encouraged to fill one ten frame first by using three of the blue counters. This helps pupils 'visualise' the answer as one ten and a two; 12. Written responses are encouraged, so that learners can link to the abstract method.

Counting on and back using a number line allows learners to visualise their thinking.

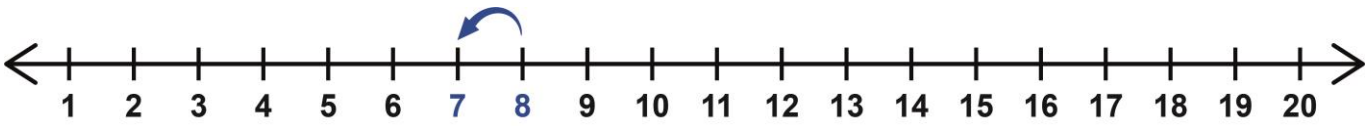
Example (1)

I have five pizzas for the birthday party and I order one more. How many pizzas do I have in total?



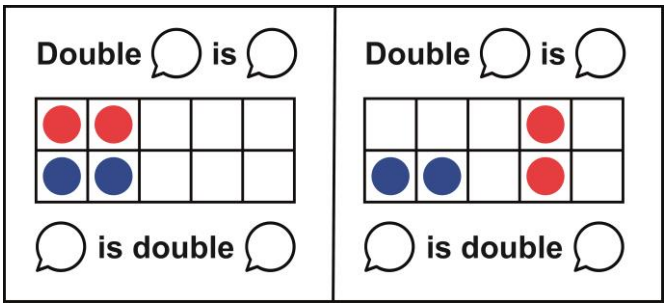
Example (2)

I have eight eggs, but I use one for my breakfast. How many eggs do I have left?



Early addition facts lead on to doubles and multiplication. It is important to progress to showing this in different ways.

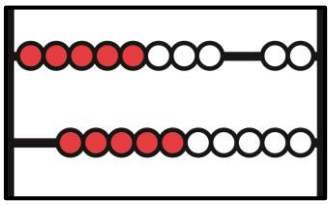
For example: What do you see?



- " I see 2 red and 2 blue"
- " I see 2 and 2 more"
- " I can represent this as 2 + 2"
- " I see 2 lots of 2"
- " I see double 2"
- " I can represent this as 2 x 2"
- " I see 4"

For example:

- "What do you see on row 1?"
- "What do you see on row 2?"
- How many more white beads are needed to make 10 on the top row?
- 5 + ? = 8
- 10 – 2 = ?



Partitioning Quantities

At early level it is important that learners are provided with opportunities to further develop their understanding of doubles and halves. Opportunities for partitioning quantities into two or more parts are also essential skills that should be developed at this stage. Formal teaching of multiplication and division is introduced in first level, however, these skills form the foundations of understanding of multiplication and division.

This should be explored using real life objects and materials that can be found both indoor and outdoor. Examples of objects and materials which could be used can be found on page five of this document.

There are three of us here today and we have nine sticks. How can we make sure we all have the same number of sticks?

We have six leaves. If we share them equally between us I wonder how many leaves we will have each?

I wonder how we could make sure that you and your friend both have an equal amount of toy cars?

Grouping and Sharing

Through grouping and sharing small quantities, learners begin to lay the foundations for multiplication and division, doubling numbers and quantities. They can make connections between arrays and number patterns.

In equal sharing learners would start with the number of groups and share the objects equally between each group.

I wonder what the best way to share these strawberries between you and your friend is?

In grouping, learners know the number of objects each group should receive. They also know what the total number of objects to begin with is.

We have ten leaves. I wonder if we can make groups of two with them?

There are nine of us here today. Can we get into groups of three?

Points to consider:

- The link between addition and subtraction should be continually reinforced to ensure that learners develop confidence and fluency in both processes.
- Saying “we can’t do 3 - 5” can cause misconceptions when learners start to work in negative numbers.
- Some learners may still count in ones to confirm ‘how many’ there are in a collection of equal groups.
- Doubling and halving should be taught together to reinforce their connection.

Reflective questions:

- **Do the available experiences and spaces (both outdoor and indoor) provide open-ended possibilities which motivate learners to explore and investigate number and number processes?**
- **Do we take time to observe the actions, emotions and words of the learners and respond sensitively to them?**

First Level

The table below includes the experiences and outcomes related to 'Number and Number Processes' at first level. The experiences and outcomes should be used in the planning of learning, teaching and assessment. It is important to note that the benchmarks are designed to support teacher professional judgement in progress towards and achievement of a level. There are a range of different experiences that learners need to be exposed to before these can be achieved.

Experiences and Outcomes	Benchmarks
I have investigated how whole numbers are constructed, can understand the importance of zero within the system and can use my knowledge to explain the link between a digit, its place and its value. **MNU 1-02a** I can use addition, subtraction, multiplication and division when solving problems, making best use of the mental strategies and written skills I have developed. **MNU 1-03a**	• Reads, writes, orders and recites whole numbers to 1000, starting from any number in the sequence. • Demonstrates understanding of zero as a placeholder in whole numbers to 1000. • Uses correct mathematical vocabulary when discussing the four operations including, subtract, add, sum of, total, multiply, product, divide and shared equally. Identifies the value of each digit in a whole number with three digits, for example, 867 = 800 + 60 + 7. • Counts forwards and backwards in 2s, 5s, 10s and 100s. • Demonstrates understanding of the commutative law, for example, 6+3 = 3+6 or 2×4 = 4×2. • Applies strategies to determine multiplication facts, for example, repeated addition, grouping, arrays and multiplication facts. • Solves addition and subtraction problems with three digit whole numbers. • Adds and subtracts multiples of 10 or 100 to or from any whole number to 1000. • Applies strategies to determine division facts, for example, repeated subtraction, equal groups, sharing equally, arrays and multiplication facts. • Uses multiplication and division facts to solve problems within the number range 0 to 1000. • Multiplies and divides whole numbers by 10 and 100 (whole number answers only). • Applies knowledge of inverse operations (addition and subtraction; multiplication and division). • Solves two step problems.

Addition and Subtraction

It is important to establish how learners reach conclusions when faced with a simple addition/subtraction calculation. Some learners may:

- use counters or other concrete materials
- count from one
- apply the counting on strategy
- adopt a more effective strategy such as the jump strategy
- use doubles/near doubles to calculate their answer.

It is important that concrete and pictorial representations are explored, investigated and discussed before moving onto more abstract methods. Learners benefit from opportunities to talk about the strategy they have chosen and, through observations and discussions, practitioners should be able to identify gaps in learners' knowledge. This can inform planning so that learners can develop more efficient strategies. These opportunities should also allow the discussion of associated vocabulary.

It is important to plan for the mental skill of estimation alongside activities so that learners can check the reasonableness of their answers.

Effective learning and teaching approaches (concrete and visual representations)

A wide variety of concrete materials are available to support understanding of addition, subtraction and place value.

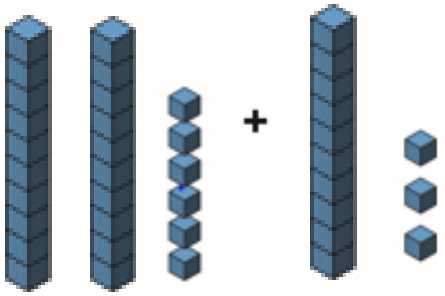
Using ten frames to find 27 + 28

Fill the third frame using counters from the last frame to make another set of ten, resulting in five sets of ten and five ones.

Using partitioning for 27 + 28:

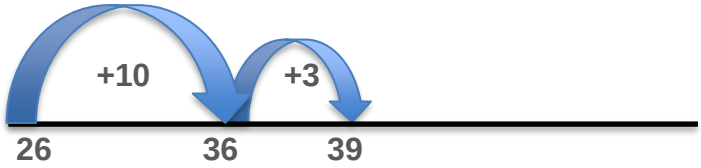
27 + 28 = 20 + 20 = 40
7 + 8 = 7 + 3 + 5
40 + 10 + 5 = 55

Using base 10 materials to find 26 + 13

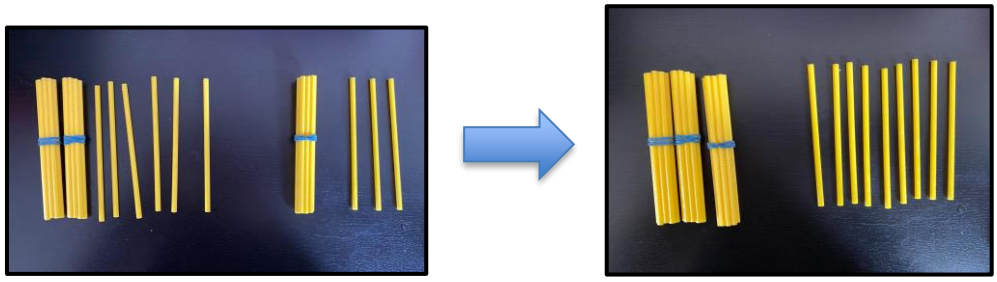


Group the tens, and then the ones together resulting in three sets of ten and nine ones.

Representing 26 + 13 on a number line:

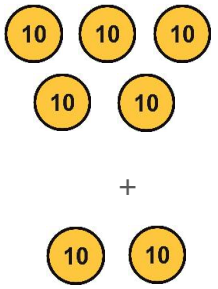
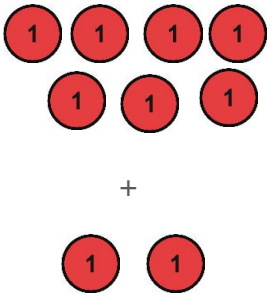


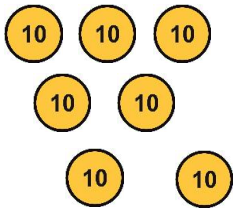
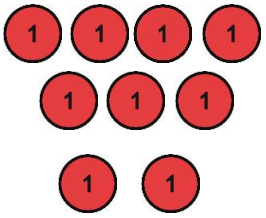
Using conceptual place value materials to find 26 + 13



Using place value counters to find 57 + 22

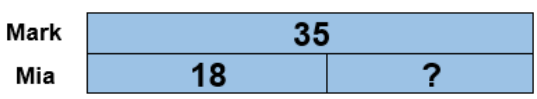
Group the tens, and then the ones together resulting in seven tens and nine ones.

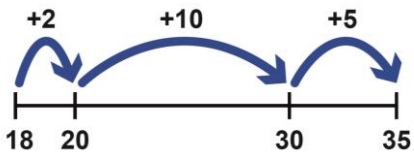
Tens	Ones
	
five tens plus two tens	seven ones plus two ones

Tens	Ones
	
seven tens	nine ones
79	

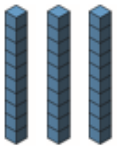
Using a variety of methods to find 35 – 18

Mark has 35 stickers. Mia has 18 stickers. How many more stickers does Mark have?

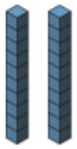
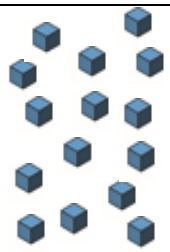




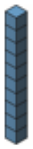
Mark has 17 more stickers than Mia.

tens	ones
	

Layout three tens and five ones.

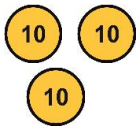
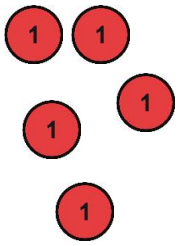
tens	ones
	

Exchange one ten for ten ones.

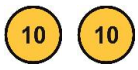
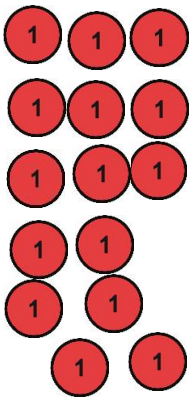
tens	ones
	

Subtract one ten and eight ones which leaves one ten and seven ones.

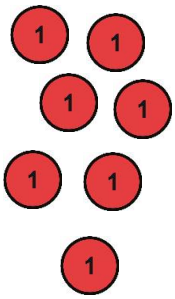
35 – 18 = 17

Tens	Ones
	

Layout three ‘ten’ counters and five ‘one’ counters.

Tens	Ones
	

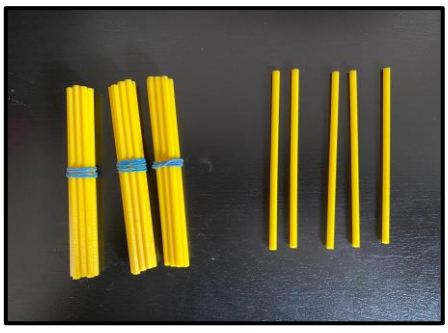
Exchange one ‘ten’ counter for ten ‘one’ counters.

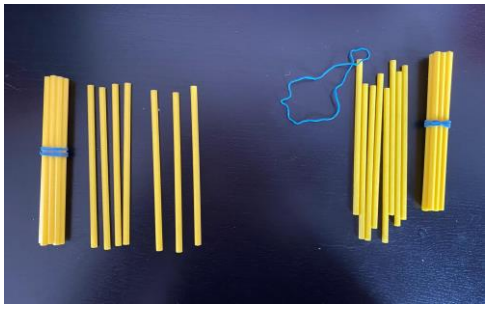
Tens	Ones
	

Subtract one ‘ten’ counter and eight ‘one’ counters which leaves one ‘ten’ and seven ‘ones’.

35 – 18 = 17

Using conceptual place value materials to find 35 – 18.





Effective learning and teaching approaches (Mental Strategies)

Learners should be exposed to a range of strategies so that they can select the most effective method for mental calculations, for example:

Counting On to find 37 + 22

Learners can use a number line or 100 square, if necessary, to count on in jumps of different sizes.

37 + 10 + 10 + 2 = 52

Partitioning to find 34 + 25

Learners can add tens and ones.

34 + 25 = 30 + 4 + 20 + 5 = 30 + 20 + 4 + 5 = 50 + 9 = 59

Compensating to find 37 + 19

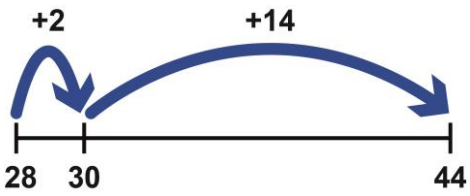
Learners can adjust one of the numbers to make the calculation easier to solve.

Add 1 to the sum, 37 + 20 = 57

Then subtract the one that was added to give the answer **56**.

Bridging to find 28 + 16

Learners can partition the 16 into 2 and 14 to count on from 30.



Reordering

When adding more numbers together, changing the order may make the calculation easier.

32 + 51 + 18 = 32 + 18 + 51 = 50 + 51 = 101

Key questions to support learning may include:

- If I know that 23 + 8 = 31, what else do I know?
- Can you convince me that twenty three add eight is thirty one?
- How many different ways can you make 30?
- Can you create a word problem using the numbers twenty and fifteen?

Points to consider:

- If the skill of estimation is well developed, learners may be able to determine the reasonableness of their solution, allowing them to be aware of making computational errors.
- Number bonds may not be secure and therefore learners may find partitioning numbers challenging.
- Learners can be moved on too quickly to sophisticated mental strategies or formal written calculations before they have developed a deep enough understanding of place value.

Reflective questions:

- **How is addition and subtraction taught at the moment?**
- **How are we ensuring that learners are using a range of strategies when performing calculations?**
- **How do we ensure learners are able to visualise numbers in different ways?**

Multiplication and Division

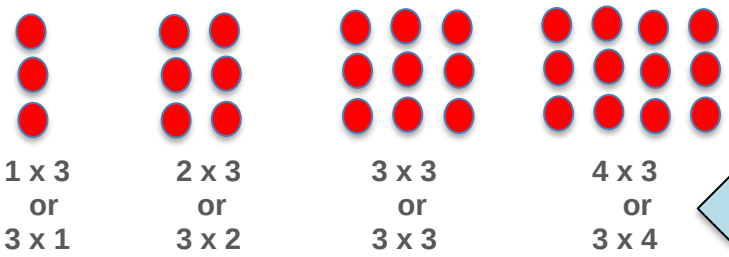
Learners should continue to investigate and model equal groups. Multiplication and division should be taught in conjunction with each other as the foundation for conceptual understanding of their inverse relationship. Learners need to understand that multiplication can be done in any order (commutative law). The use of arrays should be continued to ensure understanding through a visual approach. The understanding of times table facts should be built up; using previous knowledge of doubles, halves, counting on/back and skip counting**.**

Effective learning and teaching approaches

Using arrays to demonstrate commutativity

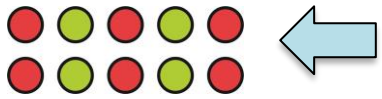
An array is a useful visual model used for multiplication. An array is formed by arranging objects into rows and columns. Each column must have an equal number of objects as the other columns, and each row must have the same number of objects as the other rows. Arrays can be useful when building up learners understanding of multiplication facts by providing a visual image.

Example of building up multiplication facts:



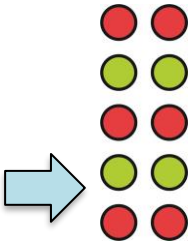
Learners should be encouraged to link multiplication and division facts. For example, when exploring this multiplication fact learners should be encouraged to also consider 12 ÷ 3 = 4 and 12 ÷ 4 = 3.

Arrays can also be used demonstrate commutativity

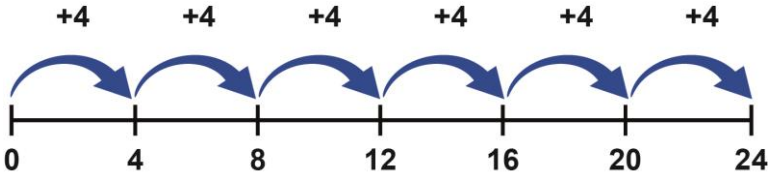


This array could be read as 2 rows of 5 or as 5 columns of 2.

By turning this representation around, the array could also show 2 rows of 5 has the same value as 5 rows of 2.
Regardless of the representation, the total of 10 is the same.



Using empty number lines to make the link with repeated addition/subtraction and multiplication/division



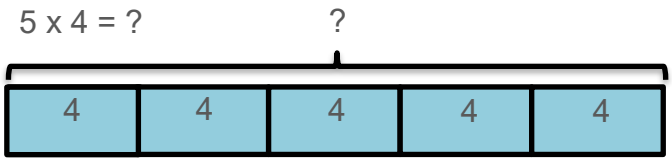
6 x 4 = 24
4 x 6 = 24
24 ÷ 6 = 4
24 ÷ 4 = 6

Using Bar Models

Bar models are a very useful way of visualising a wide variety of multiplication and division calculations.

One car has 4 wheels. How many wheels are on 5 cars?

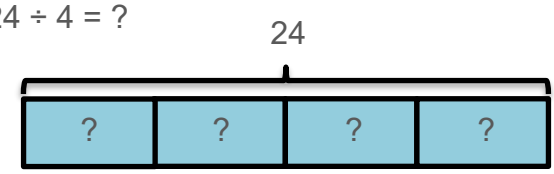
5 x 4 = ?



Learners can use skip counting or knowledge of doubles to find the answer of 20.

£24 is shared between 4 friends? How much money does each person get?

24 ÷ 4 = ?



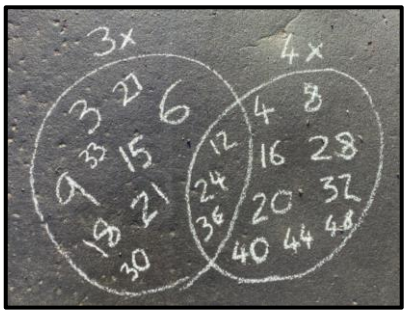
Learners might use known facts, their knowledge of halves or counting dots to calculate each share.

Outdoor learning and real life contexts

Using egg boxes to explore arrays

Using leaves to explore multiplication and division. Four lots of three leaves.

4 x 3 =12
is the same as
12 divided by 4 = 3



Drawing Venn diagrams on the playground to explore multiplication.

Sharing sticks equally.*

Creating five groups of leaves.*

*A definition of the difference between grouping and sharing can be found on page 18 of this document.

Key questions to support learning may include

- Cards come in packs of six, how many packs do I need to buy if I need 24 cards?
- I have 30 pence in my purse in 5 pence coins, how many coins do I have?

Points to consider

- Learners need to be secure in place value so that they can partition numbers in different ways (this makes division easier).
- Learners may see times tables as a list of isolated unconnected statements
- It is important that learners do not 'add a zero' when multiplying by 10, and rather they understand the change in place value for each digit.

Reflective questions

- **Is there an over reliance of rapid recall when it comes to the teaching and learning of times tables? How can this be addressed?**
- **What concrete/visual approaches are used to support understanding?**
- **How do we ensure learners are understanding the links between multiplication and division?**
- **How confident are we at teaching approaches such as using the bar model, area model and partitioning to support multiplication and division?**

Second Level

The table below includes the experiences and outcomes related to 'Number and Number Processes' at second level. The experiences and outcomes should be used in the planning of learning, teaching and assessment. It is important to note that the benchmarks are designed to support teacher professional judgement in progress towards and achievement of a level. There are a range of different experiences that learners need to be exposed to before these can be achieved.

Experiences and Outcomes	Benchmarks
I have extended the range of whole numbers I can work with and having explored how decimal fractions are constructed, can explain the link between a digit, its place and its value. **MNU 2-02a** Having determined which calculations are needed, I can solve problems involving whole numbers using a range of methods, sharing my approaches and solutions with others. **MNU 2-03a** I have explored the contexts in which problems involving decimal fractions occur and can solve related problems using a variety of methods. **MNU 2-03b** I can show my understanding of how the number line extends to include numbers less than zero and have investigated how these numbers occur and are used. **MNU 2-04a**	• Reads, writes and orders whole numbers to 1 000 000, starting from any number in the sequence. • Explains the link between a digit, its place and its value for whole numbers to 1 000 000. • Reads, writes and orders sets of decimal fractions to three decimal places. • Explains the link between a digit, its place and its value for numbers to three decimal places. • Partitions a wide range of whole numbers and decimal fractions to three decimal places, for example, 3·6 = 3 ones and 6 tenths = 36 tenths. • Adds and subtracts multiples of 10, 100 and 1000 to and from whole numbers and decimal fractions to two decimal places. • Adds and subtracts whole numbers and decimal fractions to two decimal places, within the number range 0 to 1 000 000. • Uses multiplication and division facts to the 10th multiplication table. • Multiplies and divides whole numbers by multiples of 10, 100 and 1000. • Multiplies and divides decimal fractions to two decimal places by 10, 100 and 1000. • Multiplies whole numbers by two digit numbers. • Multiplies decimal fractions to two decimal places by a single digit. • Divides whole numbers and decimal fractions to two decimal places, by a single digit, including answers expressed as decimal fractions, for example, 43 ÷ 5 = 8·6. • Identifies familiar contexts in which negative numbers are used. • Orders numbers less than zero and locates them on a number line.

Place Value

It is important that learners develop a sound understanding of place value in relation to decimal fractions. Tenths, hundredths and thousandths have a natural and familiar context in measure and money, and can be introduced in this way.

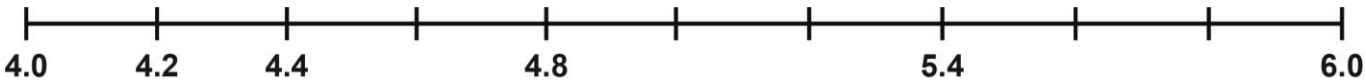
It is important that learners develop a sound understanding of place value in relation to decimal fractions. Although tenths, hundredths and thousandths have a natural and familiar context in measure and money it should be noted that using the language of money and measure, eg. £6.54 as six pounds fifty-four, can encourage learners to treat the whole number places and the decimal places as separate whole numbers, reinforcing misunderstanding. It is important that a strong abstract understanding of place value is developed alongside its practical applications.

Effective learning and teaching approaches

Rulers can provide a simple number line for counting in tenths and money provides a very familiar context for counting in hundredths. Simple problems involving counting on and counting back in these quantities, and questions involving missing numbers on a number line can help develop a number sense with decimal fractions. This can facilitate understanding of decimal addition and subtraction and allow learners to order decimal fractions based on their prior knowledge of place value.

- 6.2 cm meaning 6 centimetres and 2 millimetres where there are 10 millimetres in a centimetre, so each millimetre is one tenth of a centimetre.
- £5.04 meaning £5 and 4 pence where there are 100 pence in a pound, so each penny is one hundredth of a pound.
- 7.005 kilograms meaning 7 kilograms and 5 grams where there are 1000 grams in a kilogram so each gram is one thousandth of a kilogram.

Example:
Fill the gaps on the number line:



Rich discussion can be had around the relative value of similar decimal fractions and it is important to give learners the opportunity to discuss this and explain their thinking so any misconceptions can be addressed.

Example:
Put these decimal fractions in order:

5.8 0.58 5.08 5.008 0.508

Addition and Subtraction

Learners working at this level should be encouraged to show the many different ways they can solve problems, leading to them finding the most efficient method. Questions should be given in a problem solving context, where appropriate, allowing them to reason logically and find solutions through exploration and investigation. Learners should be supported in moving onto more formal approaches with whole numbers and decimals when a clear understanding of place value has been demonstrated.

Links to other concepts should also be explored, for example, links with money and measure, decimals, and using rounding and estimating skills to check the reasonableness of solutions.

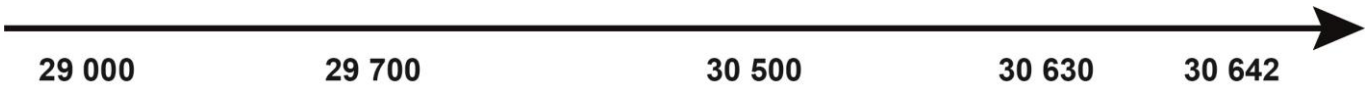
Effective Learning and Teaching Approaches

Written and algorithmic approaches may now be being used with confidence, but concrete materials or visual approaches may still be needed to ensure learners have a real understanding of the importance of place value in addition and subtraction calculations.

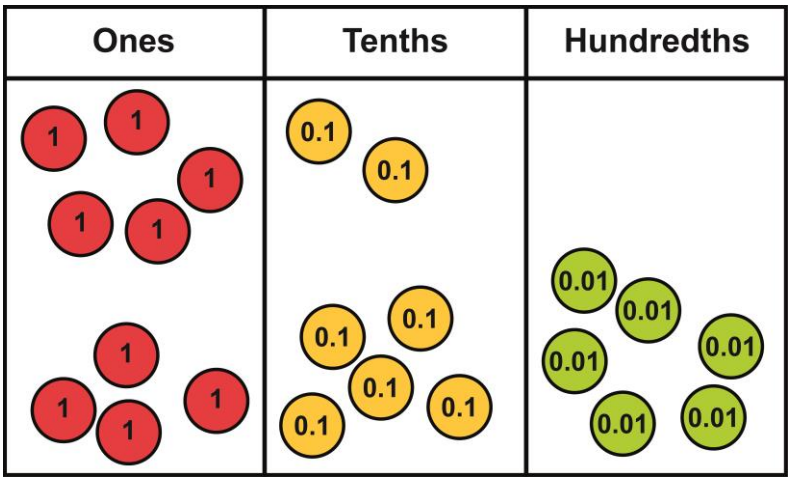
Using a partitioning approach to find 24 785 + 5 857

As the size of the numbers increase within an addition calculation, the scope for variation in learners' approaches increases, therefore learners will benefit from activities that give them the opportunity to talk about number.

Here, learners are finding 24 000 + 5000 + 700 + 800 + 80 + 50 + 5 + 7. They may group these additions in a number of ways or create a number line to help, for example:



Using place value counters to find 5.2 + 4.56



For the same calculation, learners could express both numbers to two decimal places and add ones, and hundredths:

5.20 + 4.56
5 + 4 = 9
20 hundredths plus 56 hundredths is 76 hundredths and so the answer is **9.76**

Using the counting on strategy to find £10.25 - £6.78

10.25 – 6.78 = ? becomes 6.78 + ? = 10.25

A visual approach can support the addition:

Concluding that **£10.25 - £6.78 = £3.47**

Key questions to support learning may include:

- Provide examples of column addition and subtraction with missing digits. Can learners find these digits and explain their reasoning?
- How many different ways can you solve the given calculation/problem?
  Can you think of a word problem to match the given calculation?

Points to consider:

- It is important to ensure learners have an understanding of place value for both whole numbers and decimal fractions.
- It may be unclear that 10 ones make 1 ten, 10 tens make one hundred, etc.
- Challenges may arise when using formal methods for subtraction, for example, learners being unable to split and regroup.
- It is important that time is spent developing an understanding of the value of decimal fractions.
- Saying “we can’t do 5 - 7” can cause misconceptions when learners start to work in negative numbers.
- Learners may need encouragement to use estimation skills to predict an answer.

Reflective questions:

- **How are we ensuring we are teaching for conceptual understanding?**
- **What visual approaches are our learners able to demonstrate?**
- **If learners are able to produce a written page of correct answers, can we assume that they have mastered that skill? What would we do to assess their understanding?**
- **How confident are we in the understanding of the standards and benchmarks at this level?**

Multiplication and Division

Investigation of numbers to support multiplication and division should be encouraged. A crucial skill to be encouraged and supported at this stage is to ensure pupils can partition numbers in different ways.

Effective learning and teaching approaches

In the early stages of learning, learners are encouraged to partition ten in lots of different ways to support understanding. Generally, this continues to numbers within 20, but beyond that, numbers tend to be partitioned into tens and ones only. This singular approach can make more difficult concepts harder to access. When partitioning the number 68, it is more helpful for learners to partition this beyond tens and ones. Encouraging learners to partition the number 68 in multiple ways can make the calculation 68 ÷ 4 much easier. Using multiplication facts can support this. For example, encouraging learners to look at the calculation as a multiplication, and listing known facts can help learners.

Learners could be given the calculation 4 x ? = 68. They can then list known facts:-

4 x 1 = 4
4 x 2 = 8
4 x 5 = 20
4 x 10 = 40

From these facts, can you see a way of making the number 68?

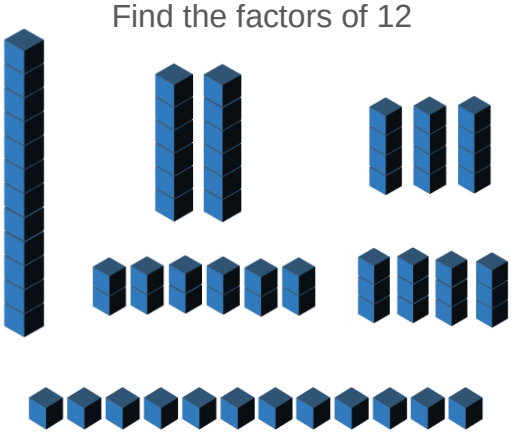
4 x 1 = 4
4 x **2** = 8
4 x **5** = 20
4 x **10** = 40

Therefore the answer is **2 + 5 + 10 = 17**.

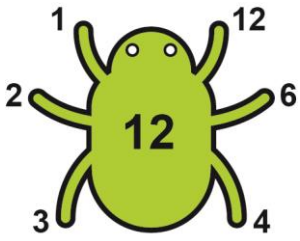
Identifying factors should be explored at this stage and can be done using a visual approach.

Example:

Find the factors of 12



represented visually



leading to

1 x 12
12 x 1

2 x 6
6 x 2

3 x 4
4 x 3

Multiplication and division by multiples of 10 is a vital skill for progression. Learners' abilities in spotting and extending patterns can be exploited here.

Example:

13 x 2 = 26
13 x 20 = ?
13 x 200 = ?
13 x 2000 = ?
130 x 2000 = ?
1300 x 2000 = ?

Extending to 1.3 x 20 = …..

Learners should be supported in using known number facts to explore alternative ways to perform a calculation. They should be provided with opportunities to share the methods they used and explain their thinking.

Example:

36 x 12 = 36 x (10 + 2) = (36 x 10) + (36 x 2) = 360 + 72 = 432
36 x 12 = (30 + 6) x 12 = (30 x 12) + (6 x 12) = 360 + 72 = 432
36 x 12 = 36 x 2 x 6 = 36 x 2 x 2 x 3 = 72 x 2 x 3 = 144 x 3 = 432
36 x 12 = (3 x 12 x 10) + (6 x 12) = 360 + 72 = 432

Using the **area model** for multiplication will develop the understanding necessary for learners before they move on to standard algorithms. It is necessary for learners to be confident in the multiplication by multiples of 10 and 100 before they move on to this stage.

Example:

To find 36 x 12

	30	6	
10	300	60	
2	60	12	300 + 60 + 60 + 12 = 432

This can be extended to multiplication of decimal fractions:

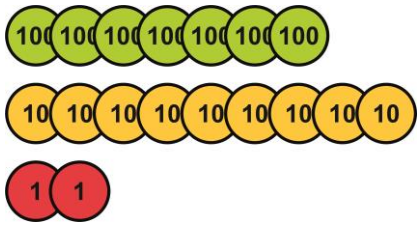
Example:

To find 3.62 x 6

	3	0.6	0.02	
6	18	3.6	0.12	18 + 3.6 + 0.12 = 21.72

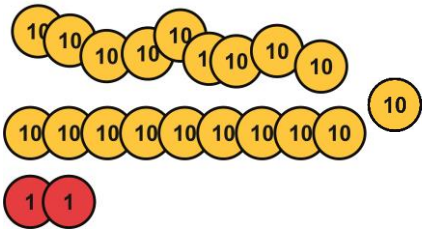
When introducing longer division calculations, it is important that learners can partition numbers into appropriate multiples here, rather than simply by place value. This can clearly demonstrate the understanding that underpins the division algorithm.

Example:
To find 792 ÷ 6 using place value counters, firstly make the number 792.

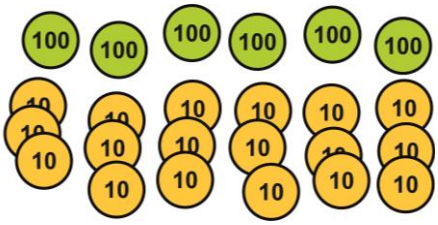


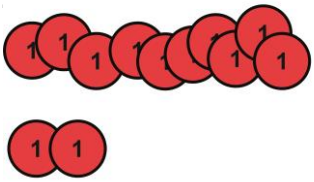
Share the hundreds equally by six. This creates a partition of 600.
Exchange the remaining hundred for tens.

792 = 600 + 192



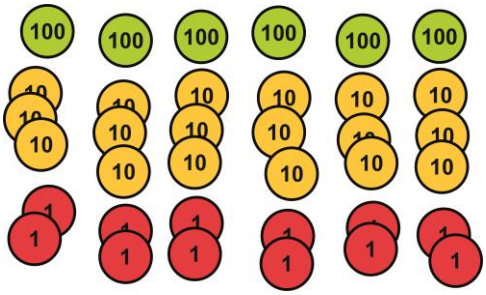
Share the tens equally by six. This creates partitions of 60.
Exchange the remaining ten for ones





792 = 600 + 60 + 60 + 60 +12

Finally, share the ones.



792 = 600 + 60 + 60 + 60 + 6 + 6
So each share is:
100 + 10 + 10 + 10 + 1 + 1 = 132

Therefore 792 ÷ 6 = 132

The area model can be used to illustrate the same calculation, by partitioning 792 into multiples of 600, 60 and 6.

6	600	192	100 + ?

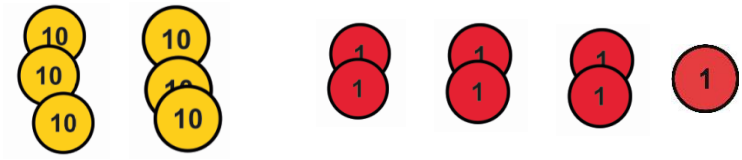
Leading to

6	600	180	12	100 + 30 + 2 = 132

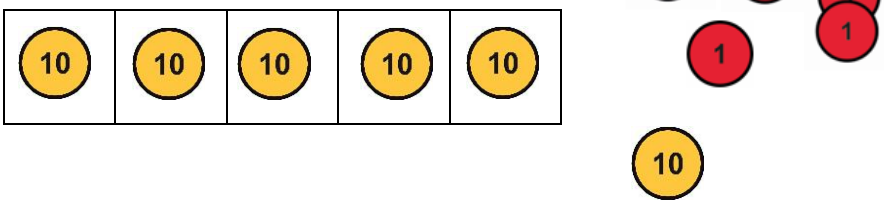
Place value counters are particularly helpful in illustrating division calculations which do not have whole number answers, and enable learners to find answers which are exact decimal fractions.

Example of division using decimal fractions.

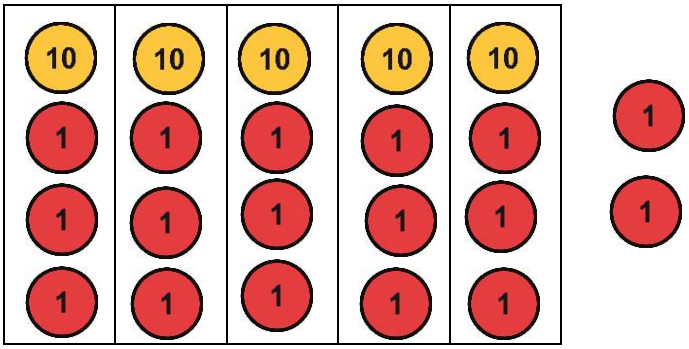
To find 67 ÷ 5, firstly create 67:



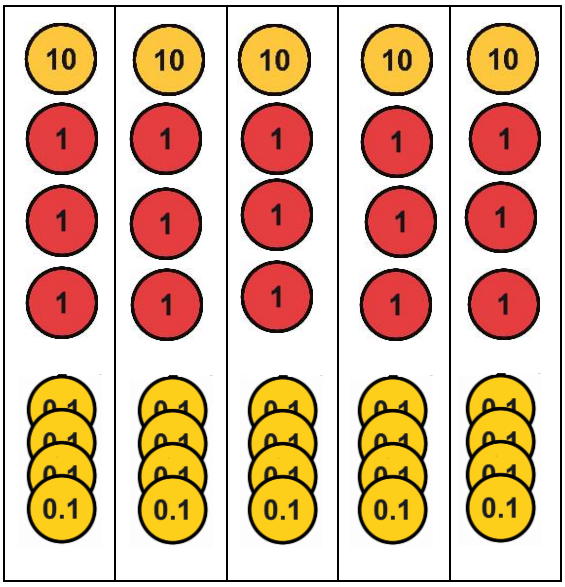
Share the tens equally between 5 as far as possible



Exchange the ten and share the ones as far as possible:



Exchange the ones for tenths and share the tenths to give the answer 13.4.



The area model can also be used to illustrate the same calculation, by partitioning 67 into multiples of 50, 5 and 0.5.

5	50	17

Leading to

5	50	15	2 = 20 tenths	10 + 3 + 0.4 = 13.4

Key questions to support learning may include:

- Which numbers, when multiplied together, give the answer 72?
- What three numbers multiply to produce 240? How many different combinations can you create?
- Is it always, sometimes or never true that an even number that is divisible by 3 is also divisible by 6?
- Can you use factor pairs to help calculate 13 x 12?

Points to consider:

- Some learners may have limited conceptual understanding of times tables.
- If multiplication and division are taught separately, learners may not develop a linked understanding of the inverse processes.
- It is important that learners understand the change in place value that results from multiplication and division by 10, 100 etc.
- Reinforce appropriate language when relevant.

Reflective questions:

- **In our experience, what challenges do learners face in developing a deep understanding of multiplication and division?**
- **Are we exposing learners to concrete and pictorial representations before moving to the abstract algorithm?**
- **How do we ensure learners are building on previously taught skills to aid the understanding of this concept i.e. partitioning numbers in different ways, doubling and halving, counting on/back?**

Third Level

The table below includes the experiences and outcomes related to 'Number and Number Processes' at third level. The experiences and outcomes should be used in the planning of learning, teaching and assessment. It is important to note that the benchmarks are designed to support teacher professional judgement in progress towards and achievement of a level. There are a range of different experiences that learners need to be exposed to before these can be achieved.

Experiences and Outcomes	Benchmarks
I can use a variety of methods to solve number problems in familiar contexts, clearly communicating my processes and solutions. **MNU 3-03a** I can continue to recall number facts quickly and use them accurately when making calculations. **MNU 3-03b** I can use my understanding of numbers less than zero to solve simple problems in context. **MNU 3-04a**	• Recalls quickly multiplication and division facts to the 10th multiplication table. • Uses multiplication and division facts to the 12th multiplication table. • Solves addition and subtraction problems working with whole numbers and decimal fractions to three decimal places. • Solves addition and subtraction problems working with integers. • Solves multiplication and division problems working with whole numbers and decimal fractions to three decimal places. • Solves multiplication and division problems working with integers.

Addition and Subtraction

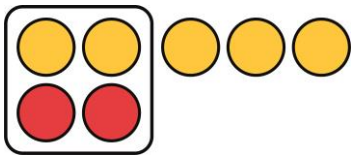
At this level, learners should demonstrate a range of methods but attempt to find and use the most efficient method depending on the calculation. Building on previous skills with concrete materials where necessary, learners can then move onto formal column calculations when they have demonstrated a clear and deep understanding of place value. Learners should be encouraged to estimate the answer to a calculation. They can then check that their answer is reasonable. Learners will benefit from opportunities to solve multi-step problems involving addition and subtraction that draw on other numeracy skills.

Effective learning and teaching approaches

Dual sided counters where the yellow side represents 1 and the red side represents -1 can be very effective in illustrating integer arithmetic.

A pair of these is called a **zero pair** and is used extensively in the arithmetic models.

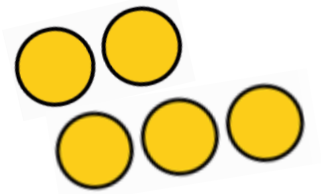
Example:
To illustrate 5 + (-2)



The two zero pairs can be removed to leave the answer of **(+)3**

Example:
To illustrate 5 – (-3)

Start with 5 (yellow counters)



Introduce 3 zero pairs (which leaves the total of 5 unchanged)

Remove the three red counters (-3) to leave the answer of **(+)8**.

Addition and subtraction of negative numbers as movement along a number line can also help learners perform calculations accurately.

Key questions to support may include:

- Convince me that (-3) – (-7) = 4.
- Show me an example of a calculation involving addition of two negative numbers and the solution (-10).

Points to consider

- Use of language such as ‘two negatives make a positive” can lead to learners developing misconceptions and making mistakes as a result. For example, learner may assume that (-3) + (-7) = 10 because there are two negatives when the correct answer is (-3) + (-7) = - 10.
- Some learners may incorrectly apply the principle of commutativity to subtraction, for example, 5 – 8 = 3.
- Ensure learners are confident and competent when entering negative numbers into their calculator and that they can successfully interpret the display.

Reflective questions

- **In what ways are we ensuring learners are ready to move onto more abstract representations?**
- **How confident are our learners on reading and writing decimal fractions and understanding the value of each digit?**
- **What concrete and visual approaches do we implement for learners to support a deep conceptual understanding?**

Multiplication and Division

Dual sided counters can develop an understanding of multiplication and division of integers. The power of patterns can also be exploited so that learners can draw conclusions and apply their understanding.

Example:

3 x 2 =	3 x (-3) =
3 x 1 =	2 x (-3) =
3 x 0 =	1 x (-3) =
3 x (-1) =	0 x (-3) =
3 x (-2) =	(-1) x (-3) =
3 x (-3) =	(-2) x (-3) =
3 x (-7) =	(-5) x (-3) =
3 x (-10) =	(-8) x (-3) =

Effective learning and teaching approaches

Learners will benefit from consolidating second level skills by solving word problems. They should be given the opportunity to think about and discuss a range of approaches to calculating their answers. The area method can be used if learners are not yet ready to move on to an algorithmic approach.

There are 24 tennis balls in a box. How many balls are there in 145 boxes?

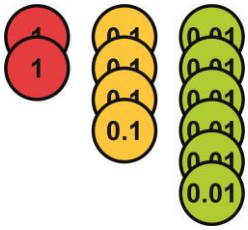
	100	40	5
20	2000	800	100
4	400	160	20

2000 + 400 + 800 + 160 + 100 + 20 = 3480

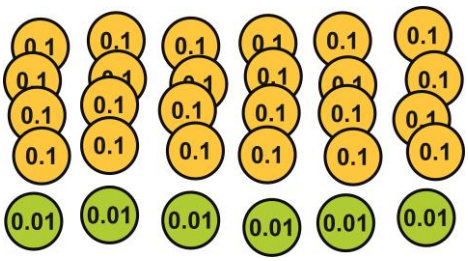
When dividing decimal fractions, strategic partitioning can enable more efficient use of place value counters or other strategies.

Example:
£38.46 is shared between 6 friends. How much money does each person receive?

£38.46 could be partitioned into £36 and £2.46 leaving £6 plus £2.46 ÷ 6 for which place value counters could be used to support learners:



Exchange the ones into tenths. Share the tenths and hundredths to give an answer of 0.41. Add this to the £6 to give a final answer of **£6.41**



At this stage, the efficiency of an algorithmic approach is likely to be attractive to learners. An area model for division can enable learners to deepen and demonstrate their understanding, however. For this calculation, learners need to partition 38.46 into multiples of 6, 0.6 and 0.06

6	36	2.46

Leading to

6	36	2.4	0.06	6 + 0.4 + 0.01 = 6.41

Key questions to support learning may include:

- (-3) and (-5) are factor pairs of a number?
  What other factor pairs can you find for that number?
- Can you use the inverse to check if the calculations provided are correct?

Points to consider:

- Not all learners are confident in explaining why numbers are added at the end of a long multiplication. Using the area model can help them visualise more clearly the steps involved.

Reflective questions:

- **How do we ensure that learners are not placing emphasis on memorising procedures but on developing their understanding about mathematical models and recognising when they fit the problem at hand?**
- **Are learners being encouraged to investigate the times tables deeply to help them discover interesting patterns and number relationships?**
- **Are we supporting and helping learners develop intuitive strategies and conceptual understanding to help master the multiplication facts?**

Fourth Level

The table below includes the experiences and outcomes related to 'Number and Number Processes' at fourth level. The experiences and outcomes should be used in the planning of learning, teaching and assessment. It is important to note that the benchmarks are designed to support teacher professional judgement in progress towards and achievement of a level. There are a range of different experiences that learners need to be exposed to before these can be achieved.

Experiences and Outcomes	Benchmarks
Having recognised similarities between new problems and problems I have solved before, I can carry out the necessary calculations to solve problems set in unfamiliar contexts. **MNU 4-03a**	• Interprets and solves multi-step problems using the four operations.

Effective learning and teaching approaches

There is no new content at fourth level in Number and Number Processes. Rather, learners should be given the opportunity to develop their procedural fluency in number processes, and to apply their knowledge by solving more complex problems in both familiar and unfamiliar contexts. There are opportunities for learners to explore the importance of their numeracy skills across the curriculum and to develop the confidence needed to apply their skills during the senior phase.

It is very important that number processing skills are used regularly to develop and maintain fluency and accuracy in all levels of calculations. Course reports for National Qualifications in mathematics subjects at all levels highlight the difficulties many candidates encounter due to a lack of precision. It is also important, that learners become confident and accurate in their use of calculators, and continue to check the reasonableness of their answers.

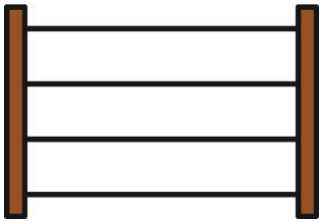
Learners benefit from the opportunity to tackle a wide range of problems which integrate number and number processes with other areas of the mathematics curriculum. The contexts of money and measure are rich sources of such problems.

Siobhan usually works 35 hours each week.
Her basic rate of pay is £10.75 per hour.
She is paid double time for any additional hours that she works.

Last week, she was paid £451.50.

Calculate the number of hours she worked last week.

A fence is made using wire and wooden posts as shown:



Posts should be spaced between 1.5 and 1.8 metres apart.
Wire costs £12 for 50 metres and wooden posts cost £4.20 each.
Klaus wants to fence a rectangular garden that is 16.8 metres by 14.5 metres.
Provide an estimate for the cost of the fence. State any assumptions you make.

Key questions to support may include:

- How can you explain your conclusions clearly?
- Are there other strategies you could have used to find your answer?
- Given the same information, what else could you work out?

Points to consider:

- Learners could first tackle problems with a calculator, and then consider how they could make the calculations without one.
- Many learners will benefit from using a visual approach to analysing the problem.
- Learners benefit from being given opportunities to tackle problems both individually and with others.

Reflective questions:

- **How do we ensure that learners can interpret questions confidently?**
- **How do we encourage learners to communicate solutions in an organised way?**
- **How do we maintain fluency and accuracy in our learners' calculations?**

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